The status of the remaining d modes cannot be determined. So it is concluded that the positive or negative definiteness of the condensed matrix using modal truncation gives the same definiteness as the complete model.

It should be noted that if the damping matrix is negative or mixed definite, the system is not stable.[1] It can be deduced that if $\xi_{ik} < -1 \quad (0 < i \leq l)$, then $\zeta_i < -1$ and there exist l negative roots in the system. The system has both oscillatory and unstable modes.

Residual Modes

In modal synthesis or substructural analysis, residual modes[7] are often used in conjunction with normal modes to compensate the truncated higher-order modes so as to increase accuracy. The higher-order mode contribution is similar to static modes. On this occasion, the derived matrices, Eqs. (10) and (11), are still applicable for two reasons. Firstly, normal modes are orthogonal to the residual modes. The orthogonality can be readily examined by the residual mode definition:

$$[\Phi_r] = [\Phi_d][\Lambda^{-1}][\Psi_d]^T[0I]^T$$

$$= ([K]^{-1} - [\Phi_k][\Lambda^{-1}][\Psi_k])[0I]^T$$

So

$$[\Psi_k]^T][M][\Phi_r] = [\Psi_k]^T[M][\Phi_d][\Lambda^{-1}][\Psi_d]^T[0I]^T$$

Because the left and right eigenvectors are orthogonal in terms of the mass matrix, i.e., $[\Psi_k]^T[M][\Phi_d] = 0$. Therefore, $[\Psi_k]^T [M][\Phi_r] = 0$. Secondly, the eigenvalues are preserved by a nonsingular transformation. So, deduction follows directly from the truncated modal transformation with only an adjustment to include the residual modes. The same forms of matrices can be obtained; but one difference is that $[\Lambda_r]$ corresponding to the residual modes is not a diagonal matrix.

Identification of Mode Order

As noted earlier, there is insufficient information to determine which mode is associated with the identified damping ratio, since a matrix is conjugate with its diagonal spectrum matrix regardless of the sequence of eigenvalues. The eigenvalues of the critical damping matrix are normally sequenced from the small to the large, not according to the mode order to which they are attributed. According to the complex mode perturbation method,[8] diagonal elements of the transformed damping matrix are roughly the same as damping ratios, with an accuracy of the first order in magnitude. So, from the position of the diagonal elements, we can determine which mode a damping ratio belongs to. Or we can simply take diagonal elements as damping ratios if they are comparatively larger than off-diagonal elements.

Concluding Remarks

This Note has presented a modal transformation method for calculating a damping ratio matrix. This matrix can be used to characterize the modal behavior of dynamic systems. The method is applicable to both nonsymmetric, as well as symmetric, systems and has been shown to require fewer computational operations than similar methods in the literature. Furthermore, a modal truncation method has been discussed which is of practical use when employing finite element methods.

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Mesh Distortion Control in Shape Optimization

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I. Introduction

IN structural shape optimization, one starts with some initial shape and iteratively changes it to minimize the objective function subject to constraints on structural response. However, as the shape changes, the finite element mesh can get badly distorted. Subsequent analysis is either unreliable or impossible. Distortion has been observed in many problems, particularly when optimizing a part that is overdesigned, in three-dimensional problems, and toward the end of a shape optimization run. Maintaining a good quality mesh does not explicitly enter into the problem formulation and consequently needs certain modifications.

Here, a mesh quality indicator based on the Jacobian is defined and implemented during the line search phase of the nonlinear programming algorithm. An element distortion parameter DP is first defined and a limiting value DP_l is chosen based on preventing degeneracy of the quadrilateral. A closed-form expression for the maximum step length along a known search direction is then derived.

Once the distortion parameter value reaches the limit, subsequent shape changes are arrested. Further shape improvements are possible only if the mesh is changed so that the distortion constraints are no longer active, which can be achieved by addition of elements, increasing the degree of the shape functions, or relocating the nodes. Here, a simple rezoning technique is used to relocate nodes.

II. Distortion Parameter DP

It is first necessary to define an indicator that represents the quality of a finite element mesh.

A distortion parameter for a general quadrilateral element is now introduced as[1]

$$DP = \frac{4 \det [J]_{\min}}{A} \tag{1}$$

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where $\det[J]_{\min}$ refers to the minimum value of the Jacobian determinant evaluated within the intervals $(-1 \le \xi \le 1, -1 \le \eta \le 1)$ and A denotes the element area. For three-dimensional hexahedral elements, we use

$$DP = \frac{8 \det[J]_{\min}}{V} \tag{2}$$

where V is the volume of the element and $\det[J]_{\min}$ is defined within the intervals $(-1 \le \xi \le 1, -1 \le \eta \le 1, -1 \le \zeta \le 1)$.

For the four-noded quadrilateral element, $\det[J]$ can be written as

$$\det[J] = c_0 + c_1\xi + c_2\eta \tag{3}$$

where c_0, c_1, and c_2 depend only on the nodal coordinates of the element. Expressions for c_0, c_1, and c_2 have been generated using the symbolic language program MAPLE. Since $\det[J]$ is a linear function in ξ and η, its minimum value will occur at one of four nodes. Let DJ_i be equal to $\det[J]$ evaluated at node i, $i = 1, \ldots, 4$. The distortion parameter in Eq. (1) is now given by

$$DP = \frac{4}{A}\min(DJ_i) \qquad i = 1, \ldots, 4 \tag{4}$$

Several properties of DP for a quadrilateral element are given as follows[2]:

1) DP is invariant with respect to translation, rotation of the coordinate system, and scaling.

2) $DP > 0$ if and only if a quadrilateral element is convex.
$DP = 0$ if and only if a quadrilateral element is degenerate (triangular).
$DP < 0$ if and only if a quadrilateral element is concave.

3) $DP \le 1.0$.
$DP = 1.0$ when an element is a parallelogram.

III. Implementing Distortion Control in Shape Optimization

In shape optimization, if X^{old} is the shape vector of nodal coordinates at the current design iteration, then a new shape is obtained from

$$X(b^o) = X_{\text{old}} + \sum_{i=1}^{ndv} b_i^0 q^i \tag{5}$$

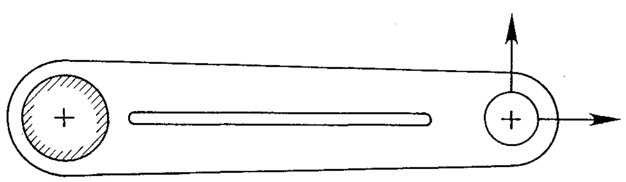

Fig. 1a Torque arm problem.

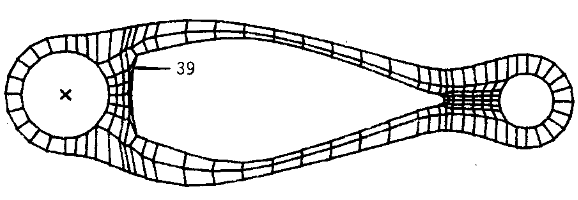

Fig. 1b Optimum shape without rezoning.

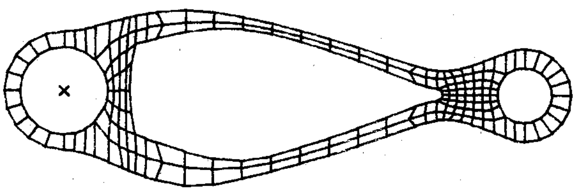

Fig. 1c Optimum shape with rezoning.

where ndv is the number of design variables. The shape change is expressed as a linear combination of velocity fields q^i.[3] The coefficients b_i^0 are design values output by the optimizer at the current iteration. The optimizer computes b^o based on 1) solving a linearized or other approximate sub-problem involving the weight, stresses, and displacements (typically) and 2) satisfying lower and upper limits on design, $b^L \le b^o \le b^U$. The bounds b^L and b^U are chosen a priori and cannot be chosen to limit mesh distortion. It has been found to be most effective to control distortion during line search, in the optimization procedure. By this strategy, mesh distortion limits are not treated as "constraints" in the nonlinear programming method. In this phase, the design variables are controlled by a step size parameter α along a known search direction $S = [S_1, \cdots, S_{ndv}]$. Thus,

$$b = \alpha S \tag{6}$$

Note that the initial value of $b = 0$ at the start of each iteration to correspond to $X = X^{\text{old}}$. Eqs. (1) and (5) implicitly define

$$DP[X(b)] = DP[X(\alpha S)] \equiv \widetilde{DP}(\alpha) \tag{7}$$

Let DP_l denote the lower limit of $\widetilde{DP}$. Solving

$$\widetilde{DP}(\alpha) = DP_l \tag{8}$$

yields an upper limit on the step size for the element under consideration, α_{DP}^e. The Jacobian matrix can be written as

$$[J] = \begin{pmatrix} J_{11}^x + \alpha J_{11}^q & J_{12}^x + \alpha J_{12}^q \\ J_{21}^x + \alpha J_{21}^q & J_{22}^x + \alpha J_{22}^q \end{pmatrix} \tag{9}$$

where

$$J_1^N = \sum_{j=1}^{4} \frac{\partial N_j}{\partial \xi} x_j^{\text{old}}, \qquad J_{11}^q = \sum_{i=1}^{ndv} \sum_{IP}^{4} S_i \frac{\partial N_j}{\partial \xi} q_{xj}^i$$

with N_j, x_j^{old}, and q_{xj}^i being the shape functions, current nodal x coordinates, and x components of the ith velocity field vector, respectively. Equation (8) together with Eq. (9) yields

$$a\alpha^2 + b\alpha + c = 0 \tag{10}$$

where a, b, and c are known constants. The solution to Eq. (10) is denoted as α_{DP}^e. Details can be found in Refs. 4 and 5. The preceding derivation is also applicable to higher order two-dimensional isoparametric elements, except that more sampling points within the element have to be chosen to find the minimum value of the determinant of the Jacobian. Further, a cubic equation is obtained for three-dimensional elements.

Now, defining

$$\alpha_{DP} = \min_e \alpha_{DP}^e \tag{11}$$

yields the step size limit based on distortion. Thus, in the design process, the best step size α is obtained from

$$\alpha = \min_e [\alpha_{DP}, \alpha_{NLP}] \tag{12}$$

where α_{NLP} is the conventional step size defined in the nonlinear programming code (here, it is the method of feasible directions).

IV. Mesh Rezoning

If the shape optimization process terminates with the step size $\alpha = \alpha_{DP}$, then the question naturally arises as to whether the final shape is a premature optimum or not. If the mesh is changed so that DP for each element is greater than the limiting value of DP_l, then the shape optimization process can be restarted. Here a rezoning technique is used that involves relocating nodes in the finite element model. Let us consider a

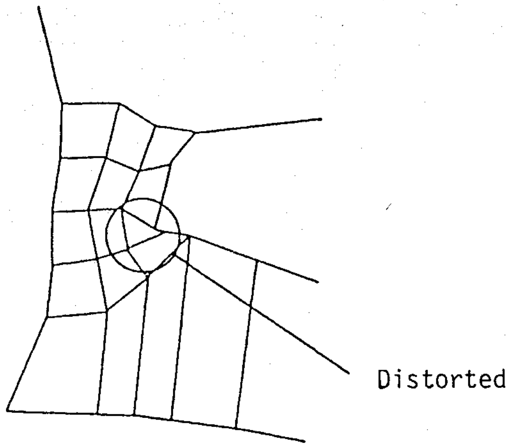

Fig. 2 Element distortion occurring in torque arm problem without distortion control.

group of four quadrilateral elements connected to a common node i. The location of node i is (x_i, y_i) and ck represents the centroid of the kth element in the group, $k = 1, \ldots, 4$. A method based on averaging a mesh quality indicator over four neighboring elements is used.[6] The formula is

$$x_i^{\text{new}} = \frac{\Sigma_{k=1}^4 DP_k x_{ck}}{\Sigma_{k=1}^4 DP_k}, \qquad y_i^{\text{new}} = \frac{\Sigma_{k=1}^4 DP_k y_{ck}}{\Sigma_{k=1}^4 DP_k} \tag{13}$$

where DP_k is the distortion parameter in element k. The rezoning procedure starts with finding all nodes that are shared by four common elements. Then Eq. (13) is used to relocate node i and this relocation is repeated until all such nodes are scanned.

V. Test Problem

The following problem is solved to illustrate the effectiveness of the mesh distortion control procedure just discussed. Weight minimization of the part shown in Fig. 1a is considered subject to element von Mises stress constraints. $DP_l = 0.2$. The step size during line search is controlled by Eq. (12). MSC/NASTRAN with a shape optimization program[5] is used. The torque arm problem is modeled using 170 CQUAD4 elements and 245 grid points. When step size is not controlled based on distortion, a concave element results (Fig. 2). The process terminates prematurely because MSC/NASTRAN does not accept this mesh. Only 5% weight reduction is gained. In contrast, with distortion control, Fig. 1b shows the final shape obtained after 17 iterations with total 38% weight reduction. However, an active distortion constraint occurs at element 39. The mesh in Fig. 1b is rezoned using Eq. (13) so that the distortion constraint is no longer active. The design process is restarted, and the final optimum shape is shown in Fig. 1c. A 42% weight reduction is achieved. We note that the impact of rezoning on weight reduction is not significant for this mesh. However, rezoning followed by optimization is found to have more impact on models with fine meshes.[4]

The concept of an active distortion constraint provides a way to check whether the true optimum shape, in the space defined by the velocity fields, is obtained. When the design iterations terminate, if there is any active distortion constraint ($\alpha = \alpha_{DP}$), then mesh refinement may yield more weight reduction. On the other hand, if $\alpha < \alpha_{DP}$ when the process terminates, it implies that a true optimum shape has been achieved.

VI. Summary

A method for explicitly controlling mesh distortion as the shape of the body changes is given. A distortion parameter for isoparametric elements is defined to avoid degeneracy of the element. The step size during the line search phase in the shape optimization process is determined so that this bound is not violated. Mesh rezoning is shown to be effective in conjunction with distortion control.

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